NOTES ON MATRIX THEORY – VI

Richard Bellman, Irving Glicksberg
and Oliver Gross

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SUMMARY

~~We derive~~ an identity in matrix theory is derived which yields a number of interesting inequalities.

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1. Introduction

In two recent notes, [1], [2], we have shown how various results relating to the determinant of A, a positive definite matrix, could be derived from the well-known identity

$$(1) \quad \frac{c_n}{|A|^{1/2}} = \int_{-\infty}^{\infty} e^{-(x,Ax)} dV_n$$

where the integration is over all x and $c_n = \sqrt{\pi}^n$, a constant depending only upon the dimension n.

If we define, for $k = 1,2,\ldots,n$,

$$(2) \quad |A|_k = \prod_{i=1}^{k} \lambda_i$$

where λ_i, $i = 1,2,\ldots,n$, are the characteristic roots of A, a positive definite matrix, arranged in increasing order of magnitude, it was shown by Ky Fan, [3], [4], that

$$(3) \quad |A\lambda + B\mu|_k \geq |A|_k^{\lambda} \, |B|_k^{\mu}, \qquad \lambda,\mu \geq 0, \qquad \lambda + \mu = 1.$$

This result, together with some additional results was recently obtained, in a different way, by Oppenheim, [5].

The purpose of the present note is to establish an identity for $|A|_k$ similar to (1). This identity may then be utilized to derive (3)

in the same manner that (1) was used to obtain the particular case $k = n$ of (3), cf. |1|, namely by a direct application of Holder's inequality.

The identity is

Theorem.

(4) $$\frac{\sqrt{\pi}^k}{|A|_k^{1/2}} = \operatorname*{Max}_L \int_L e^{-(x,Ax)} dV_k \, ,$$

where L is a k-dimensional subspace of the N-dimensional space.

Proof: It is easy to see that we may begin with the case where A is already in diagonal form, $(x,Ax) = \lambda_1 x_1^2 + \lambda_2 x_2^2 + \ldots + \lambda_n x_n^2$, since an orthogonal transformation leaves the formulation of the problem invariant. Hence we must show that

(5) $$\frac{\sqrt{\pi}^k}{(\lambda_1 \lambda_2 \ldots \lambda_k)^{1/2}} = \operatorname*{Max}_L \int_L e^{-\lambda_1 x_1^2 - \lambda_2 x_2^2 \ldots - \lambda_n x_n^2} dV_k \, .$$

Define $V_\alpha(\rho)$ to be the volume contained in the k-dimensional region

(6) $$\lambda_1 x_1^2 + \lambda_2 x_2^2 + \ldots + \lambda_n x_n^2 \leq \rho ,$$

$$(x, \alpha_i) = 0, \; i = 1,2,\ldots,(n-k) \, .$$

Then

(7) $$V_\alpha(\rho) = \rho^{k/2} V_\alpha(1)$$

and we have

$$(8) \quad \int_{(x,\alpha_1)=0} e^{-\lambda_1 x_1^2 - \lambda_2 x_2^2 - \cdots - \lambda_n x_n^2} dV_k = \int_0^\infty e^{-\rho}\, dV_\alpha(\rho)$$

$$= [\Gamma(k) \int_0^\infty e^{-\rho} \rho^{k/2 - 1} d\rho\,] V_\alpha(1).$$

To complete the proof we must show that the maximum of $V_\alpha(1)$ is attained when the relations $(x,\alpha_1) = 0$ are $x_{k+1} = x_{k+2} = \ldots = x_n = 0$. This, however, is an immediate consequence of the formula for the volume of an ellipsoid in terms of the characteristic values of the associated symmetric matrix, and the separation theorem for characteristic roots furnished by the min-max characterization of the characteristic roots.

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